Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational Physics 224 (2007) 352–364
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

# Linearized acoustic perturbation equations for low Mach number flow with variable density and temperature

Claus-Dieter Munz [a,*], Michael Dumbser [a], Sabine Roller [b]

[a] University of Stuttgart, Institute of Aerodynamics and Gasdynamics, Pfaffenwaldring 21, 70550 Stuttgart, Germany
[b] High Performance Computing Center Stuttgart (HLRS), Nobelstr. 19, 70550 Stuttgart, Germany

Received 7 September 2006; received in revised form 15 February 2007; accepted 16 February 2007
Available online 7 March 2007

Dedicated to Professor Pieter Wesseling on the occasion of his 65th birthday.

**Abstract**

When the Mach number tends to zero the compressible Navier–Stokes equations converge to the incompressible Navier–Stokes equations, under the restrictions of constant density, constant temperature and no compression from the boundary. This is a singular limit in which the pressure of the compressible equations converges at leading order to a constant thermodynamic background pressure, while a hydrodynamic pressure term appears in the incompressible equations as a Lagrangian multiplier to establish the divergence-free condition for the velocity. In this paper we consider the more general case in which variable density, variable temperature and heat transfer are present, while the Mach number is small. We discuss first the limit equations for this case, when the Mach number tends to zero. The introduction of a pressure splitting into a thermodynamic and a hydrodynamic part allows the extension of numerical methods to the zero Mach number equations in these non-standard situations. The solution of these equations is then used as the state of expansion extending the expansion about incompressible flow proposed by Hardin and Pope [J.C. Hardin, D.S. Pope, An acoustic/viscous splitting technique for computational aeroacoustics, Theor. Comput. Fluid Dyn. 6 (1995) 323–340]. The resulting linearized equations state a mathematical model for the generation and propagation of acoustic waves in this more general low Mach number regime and may be used within a hybrid aeroacoustic approach.


*Keywords:* Low Mach number flow; Linearized acoustic perturbation equations; Multiple pressure variables

## 1. Introduction

The incompressible limit of a compressible flow is rather subtle due to the fact that the propagation rate of the pressure waves becomes infinite and the equations change their type. Within this limit the pressure splits up into a thermodynamic pressure term and a hydrodynamic pressure term. If the limit solution has constant

* Corresponding author. Tel.: +49 711 685 63433; fax: +49 711 685 63402.
*E-mail addresses:* claus-dieter.munz@iag.uni-stuttgart.de (C.-D. Munz), michael.dumbser@iag.uni-stuttgart.de (M. Dumbser), roller@hlrs.de (S. Roller).

0021-9991/$ - see front matter 
doi:10.1016/j.jcp.2007.02.022

temperature and density and if the boundary values satisfy the incompressibility constraint, then the thermodynamic pressure becomes the background pressure being constant in space and time. The hydrodynamic pressure appears in the incompressible equations as a sort of a Lagrangian multiplier with no connection to the equation of state. The asymptotic analysis of Klainerman and Majda in [9,10] gives insight into this limit behavior. They gave a mathematically rigorous derivation in the isentropic case. The asymptotic analysis was formally extended by Klein to the non-isentropic case and to multiple space scales in [11] in which he also gave an overview about other asymptotic considerations in this low Mach number regime. A detailed discussion of the incompressible limit is also given in the book of Wesseling [20].

Numerical methods for the compressible equations may have difficulties with the zero Mach number limit because in the limit the speed of sound waves becomes infinite compared to the flow speed and thus leads to an elliptic coupling of pressure and velocity. Hence, all explicit numerical schemes become quite inefficient in the low Mach number regime due to their stability restriction (CFL condition). The other difficulty is that the pressure in the compressible equations converges to the thermodynamic pressure, which becomes the constant background pressure in the incompressible limit. This is the way how the equation of state for compressible flow is automatically satisfied and does not appear in the incompressible equations. For the compressible equations Bijl and Wesseling [2] introduced a splitting of the pressure into a thermodynamic and a hydrodynamic pressure term. Then they proposed an implicit numerical method that remains stable without reference to the sound velocity and which approximates the incompressible equations for Mach number zero. The constant thermodynamic pressure satisfies the equation of state and the hydrodynamic pressure serves as a Lagrangian multiplier to get the divergence-free property of the velocity. A formulation in conservative variables was later given in [18,20]. Similar to this approach Klein and Munz [12] and Munz et al. [13] proposed the multiple pressure variable (MPV) method based on the asymptotic results of Klein [11].

In [8] Hardin and Pope proposed a hydrodynamic/acoustic splitting for computational aeroacoustics which is called expansion about incompressible flow with short-hand notation EIF. Perturbation equations for the acoustics are derived by splitting up the compressible solution in an incompressible part and acoustic fluctuations:

$$\rho = \rho_0 + \rho_1 + \rho', \tag{1}$$

$$\mathbf{u} = \mathbf{u}_{\text{inc}} + \mathbf{u}', \tag{2}$$

$$p = p_{\text{inc}} + p'. \tag{3}$$

The primitive variables in the compressible equations are split into the incompressible solution identified by the index ‘inc’ and into the primed acoustic variables. The term $\rho_1$ denotes the density change due to the hydrodynamic pressure. In [8] the term $\rho_0 + \rho_1$ is called the corrected incompressible density. These relations are substituted into the compressible equations to get evolution equations for the acoustic variables. The source term is determined by the incompressible solution.

The idea of this perturbation method is to solve the flow field based on the incompressible Navier–Stokes equations and to get the noise generation and propagation in a second step via the perturbation equations. This method is attractive, if the influence of the acoustic waves to the movement of the flow can be neglected, because the solution of the incompressible Navier–Stokes equations needs less computational effort than solving their compressible counterpart. Another reason is that the acoustics waves at low Mach number have small amplitudes and large wave length compared to small structures and strong gradients in the fluid flow. The inherent numerical dissipation of a flow solver which is necessary to resolve the local gradients within the flow could destroy these waves in a short time and make a simulation up to the near far field impossible. Several subsequent contributions extended and improved this approach, see e.g., Shen and Sorensen [16], Slimon et al. [17], Bailly et al. [1], Ewert and Schröder [7] and recently Seo and Moon [15].

In this paper we extend the EIF approach for low Mach number flow with variable density, temperature gradients and heat conduction or with compression from the boundary. This extension is based on two building blocks. The incompressible solution as the state of the expansion is replaced by the solution at zero Mach number which takes into account the additional physical effects as variable density, temperature gradients, and outer compression. The flow equations in this flow regime are obtained by using the low Mach number asymptotic results as given in [9,11]. Also motivated by the asymptotic analysis the perturbation ansatz (1)–(3) is scaled with powers of the Mach number. This scaling leads to linearized acoustic equations in a straightforward way by neglecting terms of higher orders in the Mach number and also identifies the main parts of the source terms.

The outline of the paper is as follows. In Section 2 we first introduce the dimensionless compressible Navier–Stokes equations and survey the properties of the equations in the Mach number zero limit. The acoustic perturbation equations are then derived in Section 3. Various numerical results for an example at very low Mach number with compression from the boundary are presented in Section 4. In Section 5 we give some conclusions.

## 2. Governing equations

The conservation equations for mass, momentum, and total energy with viscous effects, gravitational force, and heat conduction in the conservative form are given as

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{u}) = 0, \tag{4}$$

$$\frac{\partial (\varrho \mathbf{u})}{\partial t} + \nabla \cdot [(\varrho \mathbf{u}) \circ \mathbf{u}] + \frac{1}{M^2} \nabla p = \frac{1}{Re} \nabla \cdot \mathscr{T}_{\mathrm{e}} - \frac{\rho}{Fr^2} \mathbf{e}_z, \tag{5}$$

$$\frac{\partial}{\partial t}\left[p + (\gamma - 1) M^2 \frac{\varrho \mathbf{u}^2}{2} + (\gamma - 1)\frac{M^2}{Fr^2}\varrho z\right] + \nabla \cdot \left\{\left[\gamma p + (\gamma - 1) M^2 \frac{\varrho \mathbf{u}^2}{2} + (\gamma - 1)\frac{M^2}{Fr^2}\varrho z\right]\mathbf{u}\right\}$$
$$= \frac{\gamma}{Pr \cdot Re} \nabla \cdot (\lambda \nabla T) + (\gamma - 1)\frac{M^2}{Re} \nabla \cdot (\mathscr{T}_{\mathrm{e}} \mathbf{u}), \tag{6}$$

where $\varrho$ denotes the density, $p$ the pressure, $\mathbf{u}$ the velocity, $T$ the temperature, $\eta$ the viscosity and $\lambda$ is the thermal conductivity.

Here, the equation of state for a perfect gas was used, relating the pressure to the density and the internal energy

$$p = (\gamma - 1)\varrho \varepsilon = \varrho T. \tag{7}$$

The viscous stress tensor $\mathscr{T}_{\mathrm{e}}$ for a Newtonian fluid is

$$\mathscr{T}_{\mathrm{e}} = 2\eta \mathscr{D} - \frac{2}{3}\eta (\nabla \cdot \mathbf{u}) \mathscr{I} \quad \text{with } \mathscr{D} = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}\right] \tag{8}$$

and with the identity matrix $\mathscr{I}$. The total energy $e$ per unit volume consists of the internal, the kinetic and the potential part as

$$e = \varrho \varepsilon + \frac{M^2}{2}\varrho \mathbf{u}^2 + \frac{M^2}{Fr^2}\varrho z = e_{\mathrm{i}} + M^2 e_{\mathrm{k}} + \frac{M^2}{Fr^2} e_{\mathrm{p}}. \tag{9}$$

In this dimensionless formulation we have chosen different reference values for the velocities: $u_{\mathrm{ref}}$ for the flow velocity $\mathbf{u}$ and $c_{\mathrm{ref}}$ for the sound velocity $c$. This is favorable in the low Mach number regime where these scales separate. The reference $c_{\mathrm{ref}}$ for the speed of sound is determined as $c_{\mathrm{ref}} = \sqrt{p_{\mathrm{ref}}/\rho_{\mathrm{ref}}}$ from the references for pressure and density, for simplicity we omit $\gamma$. The references $p_{\mathrm{ref}}$ and $\rho_{\mathrm{ref}}$ are typically the background values, $u_{\mathrm{ref}}$ might be an average velocity or the free stream velocity. In the flow regime, considered in this paper, the values $u_{\mathrm{ref}}$ and $c_{\mathrm{ref}}$ differ by orders of magnitudes which means that the global Mach number $M = \frac{u_{\mathrm{ref}}}{c_{\mathrm{ref}}}$ is small. The other dimensionless characteristic numbers, Reynolds number $Re$, Prandtl number $Pr$ and Froude number $Fr$, are defined as usual

$$Re = \frac{u_{\mathrm{ref}} l_{\mathrm{ref}}}{\nu_{\mathrm{ref}}}, \quad Pr = \frac{\nu_{\mathrm{ref}}}{\kappa_{\mathrm{ref}}} \quad \text{and} \quad Fr^2 = \frac{u_{\mathrm{ref}}^2}{g l_{\mathrm{ref}}}, \tag{10}$$

where $l_{\mathrm{ref}}$, $\nu_{\mathrm{ref}}$, $\kappa_{\mathrm{ref}}$ and $g$ denote the references of length, of kinematic viscosity, of thermal diffusion, and the constant of gravity, respectively.

It was rigorously proven by Klainerman and Majda [10] for isentropic flow and under some assumptions with respect to the initial data that for $M \to 0$ the compressible equations converge towards their incompressible counterparts. This is a singular limit, since the mathematical equations change their type: while in the inviscid case the compressible equations are hyperbolic or hyperbolic–parabolic in the viscous case, the incompressible equations are hyperbolic–elliptic or parabolic–elliptic. This is due to the fact that the velocity of the pressure waves

$$|\mathbf{u}| + \frac{c}{M} \tag{11}$$

for the system (4)–(6) becomes infinite. This coincides well with the physics: due to the fast pressure waves a fast pressure equalization takes place and the pressure becomes nearly constant. Thus, the flow itself can not longer generate density variations, and the flow becomes incompressible.

The incompressible limit equations as the incompressible Navier–Stokes equations are usually used as the mathematical model for fluid flow where the reference of sound velocity is some orders of magnitude larger than the reference of the flow velocity and the flow has nearly constant density and temperature. Klainerman and Majda [10] used an asymptotic expansion in powers of the Mach number to get insight into the limit behavior. The assumptions made guarantee that the solution of the limit equations has constant density and constant temperature. By a formal extension of the asymptotic analysis Klein showed in [11] that temperature and density gradients and heat conduction may also be taken into account. We will shortly describe the main facts as we need them in the following. Here, we consider the distinguished limit, when $M$ tends to zero as the sound velocity becomes infinite with respect to the fluid velocity, while the other characteristic numbers are not affected.

The factor $1/M^2$ in the momentum equation (5) in front of the pressure gradient shows that the limit $M \to 0$ is a singular one. An elliptic behavior appears that is related to the fast running pressure waves in low Mach number flows. The term $\frac{1}{M^2}\nabla p$ has to remain bounded within this limit. This is achieved when the pressure splits into a thermodynamic part constant in space and a hydrodynamic part. In the following we formally introduce into the compressible flow equations such a splitting of the pressure $p = p(\mathbf{x}, \mathbf{t})$ according to

$$p(\mathbf{x}, t) = p_0(t) + M^2 p_2(\mathbf{x}, t) \tag{12}$$

with

$$p_0(t) := \frac{1}{|\Omega|} \int_\Omega p(\mathbf{x}, t)\mathrm{d}V. \tag{13}$$

This pressure decomposition is inserted into the basic Eqs. (4)–(6) to give

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \mathbf{u}) = 0, \tag{14}$$

$$\frac{\partial (\varrho \mathbf{u})}{\partial t} + \nabla \cdot [(\varrho \mathbf{u}) \circ \mathbf{u}] + \nabla p_2 = \frac{1}{Re} \nabla \cdot \mathscr{T}_\mathrm{e} - \frac{\varrho}{Fr^2} \mathbf{e}_z, \tag{15}$$

$$\begin{aligned} M^2 \frac{\partial p_2}{\partial t} + \nabla \cdot (\gamma p \mathbf{u}) = {} & -\frac{\mathrm{d}p_0}{\mathrm{d}t} + \frac{\gamma}{Pr \cdot Re} \nabla \cdot (\lambda \nabla T) + (\gamma - 1) \frac{M^2}{Re} \nabla \cdot (\mathscr{T}_\mathrm{e} \mathbf{u}) \\ & -(\gamma - 1) M^2 \left[ \frac{\partial e_\mathrm{k}}{\partial t} + \nabla \cdot (e_\mathrm{k} \mathbf{u}) \right] - (\gamma - 1) \frac{M^2}{Fr^2} \left[ \frac{\partial e_\mathrm{p}}{\partial t} + \nabla \cdot (e_\mathrm{p} \mathbf{u}) \right]. \end{aligned} \tag{16}$$

For $M > 0$ and a given pressure $p_0(t)$ this system is equivalent to the compressible Navier–Stokes equations, if $p$ in (12) satisfies the equation of state.

Let us assume for the moment that in the case $M \to 0$ the pressure term $p_0$, the density, as well as the temperature are constant. In this case, the momentum equation (15) coincides with the counterpart in the incompressible equations, in which the role of the hydrodynamic pressure is taken over by $p_2$. The energy equation (16) simplifies to $\nabla \cdot \mathbf{u} = 0$ and the density equation (14) is automatically satisfied. Hence, under these assumptions, which are the usual assumptions for incompressible flow, the system (14)–(16) formally equals to the incompressible Navier–Stokes equations. The pressure term $p_0$ is the background pressure being constant in space and time and satisfying the equation of state. The pressure $p_2$ is the hydrodynamic pressure which has nothing to do with the equation of state and is some sort of Lagrangian multiplier. According to (9) the total energy is constant and identical to the internal energy. Hence, introducing the pressure decomposition and assuming that all terms are bounded, the compressible equations formally converge to the incompressible equations, when the Mach number tends to zero.

Within the construction of numerical methods for solving the flow equations at low Mach numbers a pressure decomposition was introduced by Wesseling and his co-workers in [2] and in the multiple pressure variables approach of Klein and Munz [12] and Munz et al. in [13] in primitive variables. The extension to the conservative formulation of the flow equations as used in (14)–(16) was proposed in [18,19]. By that, the shock-capturing property can be established and a unified method for all Mach numbers can be constructed. For a detailed discussion and an overview of the other approaches to capture fluid flow at small Mach numbers see [20].

We next discuss the zero Mach number limit in the more general case. For $M \rightarrow 0$ the energy equation formally reduces to

$$\frac{\mathrm{d}p_0}{\mathrm{d}t} = -\gamma p_0 \nabla \cdot \mathbf{u} + \frac{\gamma}{Pr \cdot Re} \nabla \cdot (\lambda \nabla T). \tag{17}$$

This is an evolution equation for the leading order pressure term $p_0$. We integrate the equation over the whole computational domain $\Omega$ and use the definition of $p_0$ according to (13) to obtain

$$\begin{aligned}\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} &= \frac{1}{|\Omega|}\left[-\gamma p_0 \oint_{\partial\Omega} (\mathbf{u}\cdot\mathbf{n})\mathrm{d}A + \frac{\gamma}{Pr \cdot Re}\oint_{\partial\Omega}(\lambda\nabla T\cdot\mathbf{n})\mathrm{d}A\right] \\ &= \frac{\gamma p_0}{|\Omega|}\left\{-\oint_{\partial\Omega}(\mathbf{u}\cdot\mathbf{n})\mathrm{d}A + \frac{1}{Pr\cdot Re}\oint_{\partial\Omega}\left[\lambda\nabla\left(\frac{1}{\varrho}\right)\cdot\mathbf{n}\right]\mathrm{d}A\right\},\end{aligned} \tag{18}$$

where $\mathbf{n}$ denotes the outwards directed unit normal vector at the boundary $\partial\Omega$. Here, the volume integrals of the divergence terms were reformulated as surface integrals of the fluxes into the normal direction by applying the Gauss theorem. The equation (18) describes the global pressure change in time due to compression and heat transfer from the boundary. The global pressure $p_0$ is the thermodynamic variable and closely related to the internal energy according to Eq. (9) for $M \rightarrow 0$, so that the equation above corresponds to the first law of thermodynamics.

Eqs. (17) and (18) can be combined and yield the divergence condition

$$\nabla\cdot\mathbf{u} = \frac{1}{p_0 Pr\cdot Re}\nabla\cdot(\lambda\nabla T) + \frac{1}{|\Omega|}\left\{\oint_{\partial\Omega}(\mathbf{u}\cdot\mathbf{n})\mathrm{d}A - \frac{1}{p_0 Pr\cdot Re}\oint_{\partial\Omega}[\lambda\nabla T\cdot\mathbf{n}]\mathrm{d}A\right\}. \tag{19}$$

It is obvious, that the divergence-free condition of velocity for incompressible flow is retained under the absence of the outer compression and heat transfer. Therefore, this equation may be interpreted as a generalization of the divergence condition for more general zero Mach number flows. Eqs. (14), (15) and (19) describe zero Mach number flow with variable density, variable temperature, with heat conduction and under the influence of compression or expansion at the boundaries. The pressure $p_2$ enables to satisfy the divergence condition (19) as a Lagrangian multiplier. This system of equations may be used as a mathematical model within a flow regime in which $u_{\mathrm{ref}}$ is much smaller than $c_{\mathrm{ref}}$ with variable density, variable temperature and diffusion. The equations in this form have already been considered by Chenoweth and Paolucci in [3] to describe natural convection with large temperature differences and were called the subsonic approximation.

## 3. Expansion about zero Mach number flow

In this section we consider compressible corrections of this zero Mach number flow solution using a perturbation ansatz as proposed in [8] about incompressible flow. The solution of the zero Mach number equations is named in the following as $\rho_0, u_0, p_0, p_2, T_0$ and is given by (14), (15), (18), (19), and the thermodynamic equation of state $p_0 = \rho_0 T_0$.

Motivated by the asymptotic results we introduce a scaled perturbation approach of the form

$$\rho(\mathbf{x},t) = \rho_0(\mathbf{x},t) + M^2\rho_2(\mathbf{x},t) + M^2\rho'(\mathbf{x},t), \tag{20}$$

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u}_0(\mathbf{x},t) + M\mathbf{u}'(\mathbf{x},t), \tag{21}$$

$$p(\mathbf{x},t) = p_0(t) + M^2 p_2(\mathbf{x},t) + M^2 p'(\mathbf{x},t). \tag{22}$$

The pressure is split into a thermodynamic part $p_0$, into the hydrodynamic part $p_2$ and into the acoustic fluctuations $p'$. The density is also split into the density $\rho_0$ for $M \to 0$, into the acoustic fluctuations $\rho'$, and into $\rho_2$ which introduces a density change due to the hydrodynamic pressure. We note that $\rho_0$ may depend on $\mathbf{x}$, when density variations in the initial data or heat conduction are present. The corresponding temperature is obtained from the equation of state $p_0 = \rho_0 T_0$.

For consistency we have to satisfy with this perturbation ansatz Eqs. (20) and (21) and the equation of state (22). If it is given in the form $p = p(\rho, S)$ where $S$ denotes the entropy, then the equation

$$p_0 + M^2 p_2 + M^2 p' = p(\rho_0 + M^2 \rho_2 + M^2 \rho', S_0 + M^2 S_2) \tag{23}$$

has to be satisfied. If the truncated Taylor expansion

$$p(\rho, S) = p(\rho_0, S_0) + \frac{\partial p}{\partial \rho}(\rho_0, S_0)(\rho - \rho_0) + \frac{\partial p}{\partial S}(\rho_0, S_0)(S - S_0) \tag{24}$$

is substituted, then this relation simplifies to

$$p_2 + p' = \frac{\partial p}{\partial \rho}(\rho_0, S_0)(\rho_2 + \rho') + \frac{\partial p}{\partial S}(\rho_0, S_0) S_2. \tag{25}$$

The assumption that the acoustic perturbations propagate isentropically leads to the relation $p' = c_0^2 \rho'$ and the corresponding terms in (25) cancel out. The result is an equation for the balance of the hydrodynamic pressure with density changes and entropy changes. We neglect in (25) the second term and define

$$\rho_2 := \frac{p_2}{c_0^2} \quad \text{with } c_0^2 = \gamma \frac{p_0}{\rho_0}. \tag{26}$$

to balance the hydrodynamic part.

In the EIF approach Hardin and Pope [8] estimated the second term by a time average of the pressure. The arguments here are that the time-averaged pressure may be assumed to be a result of dissipative mechanisms, since the effects of viscosity and heat conduction are normally slow on an acoustic time scale. Their expansion was based on the incompressible solution supplemented with a temperature transport equation including heat conduction. We replaced here this incompressible solution by a $M \to 0$ solution where density, temperature, and the thermodynamic background pressure satisfy the equation of state. The leading terms of the diffusive effects and entropy changes should be captured in these leading order variables density $\rho_0$, temperature $T_0$, and thermodynamic pressure term $p_0$ and are removed from the perturbation quantities. Hence, we neglect the entropy changes at this place. Another difference to the EIF approach by Hardin and Pope [8] is that we use the scaling in powers of the Mach number $M$ as motivated by the low Mach number asymptotic analysis. Later on we use the Mach number scaling to consider the basic terms and neglect all terms which have a factor with a greater power in the Mach number. This automatically leads to a linearization and identifies the most important terms in the sources for sufficiently small Mach numbers.

We note that the introduction of the hydrodynamic density variation is not inevitable at this place but favorable. These density changes are connected with pure hydrodynamic motion. In the incompressible limit, these density changes do not appear since the equation of state is satisfied by the background pressure, the background density and the background internal energy, which are all constant in space and time. For low but non-zero Mach number flows all pressure terms have to be included into the equation of state and lead to hydrodynamic density corrections. These corrections are associated with the flow motion and are convected with fluid velocity. If these corrections are introduced explicitly in the expansion, then they are shifted from the wave operator to the right-hand side. We discuss this aspect once again in the discussion of the perturbation equations.

The relations (20)–(22) are inserted into the compressible Navier–Stokes equations (4)–(6). On the left-hand side we write all the derivatives in space and time of the primed variables to get an evolution equation for the perturbations. The right-hand side contains all the other terms and are interpreted as the sources. Higher order terms with respect to the powers of the Mach number are neglected. Doing so we obtain the system

$$\frac{\partial \rho'}{\partial t} + \mathbf{u}_0 \cdot \nabla \rho' + \frac{1}{M} \nabla \cdot (\rho_0 \mathbf{u}') = -\frac{D\rho_2}{Dt} - (\rho_2 + \rho')\nabla \cdot \mathbf{u}_0, \tag{27}$$

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}' + (\mathbf{u}' \cdot \nabla)\mathbf{u}_0 + \frac{1}{M\rho_0} \nabla p' = \frac{1}{Re\rho_0} \nabla \cdot \mathscr{T}_e(\mathbf{u}'), \tag{28}$$

$$\frac{\partial p'}{\partial t} + \mathbf{u}_0 \cdot \nabla p' + \frac{1}{M} \gamma p_0 \nabla \cdot \mathbf{u}' = -\frac{Dp_2}{Dt} - (p_2 + p')\nabla \cdot \mathbf{u}_0. + \frac{\gamma}{PrRe} \nabla \cdot \lambda \nabla T_2. \tag{29}$$

Here, we changed from the conservative formulation to the simpler evolution equations for density, velocity and pressure, respectively. As long as smooth solutions of the acoustic equations are considered the simpler formulation in primitive variables should be preferred. $D/Dt$ is the abbreviation of $\partial/\partial t + u_0 \cdot \nabla$ and $T_2$ is defined such that the equation of state is satisfied which leads to the relation $\rho_0 T_2 = p_2 + p' - (\rho_2 + \rho')T_0$. The propagation rate of the acoustic waves in this system becomes infinite while their amplitude tends to zero when the Mach number tends to zero. The mathematical justification of the linearized acoustics was given by Klainerman and Majda [10] who showed that the solutions of these equations are uniformly bounded with respect to $M$ in the isentropic case. The left-hand side of this system is usually called the linearized Euler equations (LEE).

Let us now consider the case, when all diffusion effects, compression and heat flux from the boundary are neglected and the fluid motion is isentropic. Then the linearized perturbation equations can be simplified to

$$\frac{\partial \rho'}{\partial t} + \mathbf{u}_0 \cdot \nabla \rho' + \frac{1}{M} \rho_0 \nabla \cdot \mathbf{u}' = -\frac{D\rho_2}{Dt}, \tag{30}$$

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}' + \frac{1}{M\rho_0} \nabla p' = -(\mathbf{u}' \cdot \nabla)\mathbf{u}_0, \tag{31}$$

$$\frac{\partial p'}{\partial t} + \mathbf{u}_0 \cdot \nabla p' + \frac{1}{M} \gamma p_0 \nabla \cdot \mathbf{u}' = -\frac{Dp_2}{Dt}. \tag{32}$$

The main acoustic source is the term $Dp_2/Dt$ in the pressure equation and $D\rho_2/Dt$ in the density equations.

A difficulty in the perturbation approach based on the linearized Euler equations is that they do not only describe acoustic wave motion, but also vorticity and entropy modes. Especially in flows where hydrodynamic instabilities are present as, e.g., in shear flow, fluid instabilities may also occur and increase according to the linear theory with an exponential rate and may falsify the acoustic motion. Seo and Moon [15] carefully analyzed the LEE and reformulated in the velocity equations (31) $(\mathbf{u}' \cdot \nabla)\mathbf{u}_0 + (\mathbf{u}_0 \cdot \nabla \mathbf{u}')$ into $\nabla(\mathbf{u}'\mathbf{u}_0)+$ two terms involving the vorticity of the acoustic propagation and the vortical hydrodynamic motion. Because the additional two terms are associated with the generation and transport of perturbed vorticity only, they dropped them to get the proper acoustical motion in the LPCE (linearized perturbed compressible equations). In other respects the system (30)–(32) coincides well with LPCE in the isenstropic case. Seo and Moon did not explicitly introduce the scaling with the Mach number and the explicit hydrodynamic correction of the density. Instead they used the principal perturbation variable $\rho_2 + \rho'$ and got therefore some discrepancy in the pressure equation, too. Additionally, we do not take into account the term $(\mathbf{u}' \cdot \nabla)p_2$ which is in our consideration $O(M^3)$. Another approach to get rid of the hydrodynamic instabilities is a reformulation into the APE system due to Ewert and Schröder [7]. They reformulated the wave propagation model such that vorticity and entropy modes do not appear anymore and obtained a system of equations for the acoustic modes only. They also succeeded to extend these considerations to compressible main flow. We do not consider this aspect in the following.

The impact to introduce hydrodynamic density corrections can directly be seen in this isentropic case. We multiply the equation for the acoustic density fluctuations (30) by $c_0^2$ and subtract it from the pressure equation to obtain

$$\frac{\partial}{\partial t}(p' - c_0^2\rho') + \mathbf{u}_0 \cdot \nabla(p' - c_0^2\rho') = -\frac{D}{Dt}(p_2 - c_0^2\rho_2). \tag{33}$$

Here, the source terms cancel out and the relation $p' - c_0^2\rho' =$ constant in time along the particle paths is valid. Hence, the acoustic wave propagation is fully isentropic as expected. We can either solve the density equation (30) or the pressure equation (32) together with the velocity equation (31) and the isentropic relation. If the density corrections are not introduced, then we need the full linearized Euler equations, because the hydrodynamic density corrections are part of the solution. If furthermore the convection of the acoustic perturbations can be neglected, then the whole set of equations (30)–(32) simply reduces to the linear acoustic wave equation

$$\frac{\partial^2 p'}{\partial t^2} - \frac{c_0^2}{M^2}\Delta p' = -\frac{\partial^2 p_2}{\partial t^2} \tag{34}$$

or equivalently the density equation

$$\frac{\partial^2 \rho'}{\partial t^2} - \frac{c_0^2}{M^2}\Delta \rho' = -\frac{\partial^2 \rho_2}{\partial t^2} \tag{35}$$

with $\Delta = \nabla \cdot \nabla$. Here, the advantage to decompose the $M^2$-terms of the density into $\rho_2$ from (26) and the acoustic fluctuations $\rho'$ becomes obvious again. In the density equation the compressibility effects due to the hydrodynamic motion are moved from the wave propagation on the left-hand side to the right-hand side and play the role of the source terms. Using $\rho_2 + \rho'$ as a primary variable one has to be more careful. Neglecting the fluid convection would not result in a simple wave equation for density, because the assumption of an isentropic relation between these density and pressure perturbations is not valid. Otherwise one would have a wave equation for the density perturbations without source terms. The wave equation for the density perturbations, which directly corresponds to the wave equation for the pressure, can be obtained only if the hydrodynamic density corrections are subtracted.

The inclusion of thermal conduction, variable density and compression from the boundary in (27)–(29) makes the situation more complicated. The wave operator for the wave propagations is still the same, but additional source terms on the right-hand side appear compared to the expansion about the incompressible solution This is due to the fact that here the divergence of the velocity $\nabla \cdot \mathbf{u}_0$ may not vanish. Additional source terms occur in the density and pressure equations determined by the divergence of the velocity $u_0$ and heat conduction.

## 4. A non-trivial one-dimensional example

In the following we consider an one-dimensional example with compression from the boundary at a global Mach number $M = 0.0025$. An acoustic wave is generated by the time-dependent boundary values for the velocity

$$u(a,t) = M\sin(t), \quad u(b,t) = -M\sin(t) \quad \text{for } t \in \mathrm{R}_0^+. \tag{36}$$

The equations for Mach number $M = 0$ generated by these boundary values possess a non-trivial solution, because the incompressibility constraint $u(a,t) - u(b,t) = 0$ is not satisfied. The solution can be calculated analytically. We have the following properties of the flow field: the thermodynamic pressure $p_0$ is constant in space, but a function of time, the background density $\rho_0$ is constant in space, but a function of time, the velocity $u_0$ is a function in time and a linear function in space, and the hydrodynamic pressure $p_2$ is a function in time and a quadratic function in space. According to the excitation from the boundary values (36) the thermodynamic pressure is given by (17) that simplifies in our case to

$$\frac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -\frac{\gamma p_0(t)}{|I|}(u_0(b,t) - u_0(a,t)), \tag{37}$$

where $I$ denotes the spatial interval $I = [a,b] = [0,\pi]$ and $|I| = \pi$ its length. Hence, the $x$-derivative of the velocity is given by

$$\frac{\partial u_0(x,t)}{\partial x} = -\frac{1}{\gamma p_0(t)}\frac{\mathrm{d}p_0(t)}{\mathrm{d}t}. \tag{38}$$

Because the velocity is known, the hydrodynamic pressure is obtained from the velocity equation

$$\frac{\partial u_0(x,t)}{\partial t} + u_0\frac{\partial u_0(x,t)}{\partial x} + \frac{1}{\rho_0(t)}\frac{\partial p_2(x,t)}{\partial x} = 0 \tag{39}$$

and the density from the continuity equation

$$\frac{\mathrm{d}\rho_0(t)}{\mathrm{d}t} = -\rho_0(t)\frac{\partial u_0}{\partial x}(t). \tag{40}$$

The non-trivial zero Mach number solution for $u_0$, $p_0$, $\rho_0$ and $p_2$ is then obtained analytically after integration of (37)–(40) as follows:

$$u_0(x,t) = M\sin(t)\left(1-\frac{2x}{\pi}\right), \tag{41}$$

$$p_0(t) = \frac{1}{\gamma} - 2\frac{M(\cos(t)-1)}{\pi}, \tag{42}$$

$$\rho_0(t) = A(M,t)\cdot\left(\cosh\left(2\frac{M}{\pi}\right) - \sinh\left(2\frac{M}{\pi}\right)\right)^{-1}, \tag{43}$$

$$p_2(x,t) = A(M,t)\cdot B(M)\cdot(\cos(t)\pi + M(\cos(2t)-1))\frac{x(x-\pi)}{\pi^2}, \tag{44}$$

with

$$A(M,t) = \mathrm{e}^{-\frac{2}{\pi}M\cos(t)} \quad \text{and} \quad B(M) = \sinh\left(2\frac{M}{\pi}\right) + \cosh\left(2\frac{M}{\pi}\right). \tag{45}$$

The analytical expressions given by (41)–(44) are then used to calculate the source terms on the right-hand side of the acoustic perturbation equations (27)–(29). Diffusion effects are neglected. As a reference solution, we compute the entire problem solving the full non-linear compressible Euler equations with the corresponding boundary conditions (36) for the velocity and with the initial condition $\rho(x,0)=1$, $u(x,0)=0$ and $p(x,0)=1/\gamma$ with $\gamma=1.4$ up to time $t=2.0$ using a third order ADER discontinuous Galerkin scheme [5,6] on 1000 elements.

The linearized Euler equations (27)–(29) are then solved numerically up to $t=2.0$ with the appropriate boundary and initial conditions using a third order ADER finite volume (ADER-FV) scheme [4] with 250 cells in the space interval. The total flow quantities $\rho=\rho_0+M^2(\rho_2+\rho')$, $u=u_0+Mu'$ and $p=p_0+M^2(p_2+p')$ are depicted in Fig. 1 for both the non-linear computation and the computation with the linearized Euler equations. We note an excellent agreement between the non-linear and the linearized simulation.

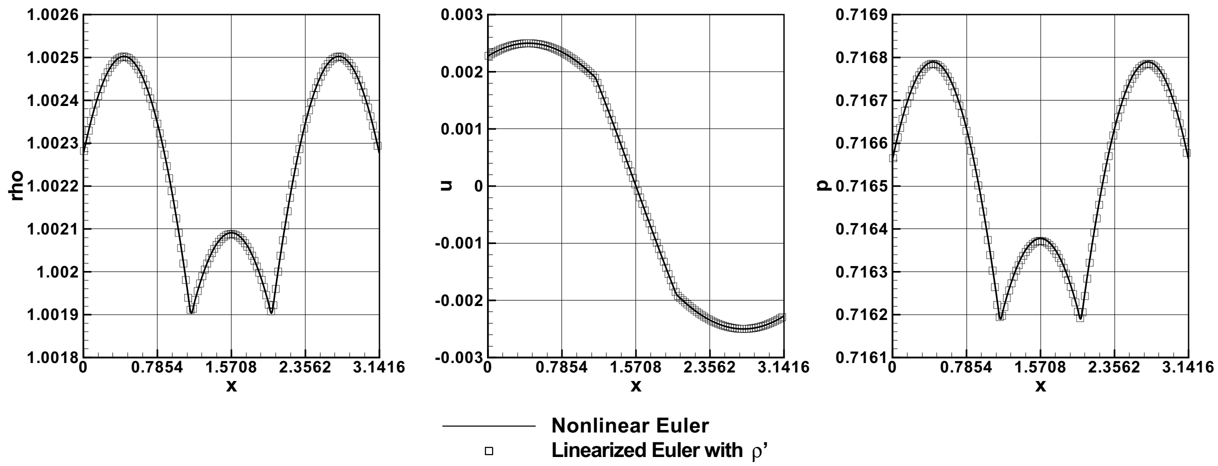


Fig. 1. The total flow quantities $\rho$, $u$ and $p$ at $t=2.0$ of the boundary value problem (36) computed with the linearized Euler equations (27)–(29) compared with the non-linear, fully compressible Euler equations.

Since in this test problem the values of the velocity are very small, the influence of the flow field onto the acoustic wave propagation may be neglected. If we set $u_0 = 0$, $\rho_0 = 1$ and $p_0 = 1/\gamma$ in the linearized Euler equations (27)–(29), then the mathematical model of the linearized Euler equations reduces to the simple inhomogeneous wave equation (34) for the acoustic pressure $p'$. The solution obtained at $t = 2.0$ for the total flow quantities $\rho$, $u$ and $p$ with this approach is depicted in Fig. 2, where again an excellent agreement of the non-linear and the linearized computation can be observed, in spite of neglecting all convection effects. The numerical method in this case was again a third order ADER-FV scheme on 250 cells.

In the following we perform an additional calculation for which we dropped the source term $-\frac{D\rho_2}{Dt}$ in the density equation (27). Hence, the density correction for the hydrodynamic pressure $p_2$ is also shifted to the left-hand side. We call the resulting density corrections $\rho'' = \rho_2 + \rho'$ and in this case the density equation (27) for $\rho'$ is replaced by the following equation for $\rho''$:

$$\frac{\partial \rho''}{\partial t} + \mathbf{u}_0 \cdot \nabla \rho'' + \frac{1}{M} \rho_0 \nabla \cdot \mathbf{u}' = -\rho'' \nabla \cdot \mathbf{u}_0. \tag{46}$$

In Fig. 3 the results obtained for the full flow quantities $\rho = \rho_0 + M^2 \rho''$, $u$ and $p$ are again compared with the reference solution obtained from the fully compressible non-linear Euler equations at $t = 2.0$. We see that also with this formulation the linearized approach is able to capture very well the behavior of the reference solution. We note that the density corrections $\rho''$ are in this case not longer purely acoustic corrections and do not satisfy the isentropic relation with the acoustic pressure-corrections $p'$ as outlined previously. Hence, it seems to be favorable for the simulation of acoustic wave propagation to shift the hydrodynamic density correction to the right-hand side.

Finally we show a comparison of all the above-mentioned linearized approaches with the reference solution in Fig. 4, where a zoom into the distribution of total density $\rho$ is plotted at $t = 2.0$. We clearly observe that both versions of the linearized Euler equations yield results that are closer to the reference solution than the pure wave equation. A physical explanation is the following: since the velocity gradient has a maximum in the middle of the computational domain $I$, we expect the convection effects to be larger in this region compared to regions closer to the boundaries of $I$. At $t = 2.0$ the boundaries $a$ and $b$ are still moving inwards, see the profile of $u$ in Figs. 1–3. In this case the convection will lead to an *increase* of density in the middle of the domain $I$. It can be clearly seen in Fig. 4 that in the considered region the density is *underestimated* by the pure wave equation due to the fact that convection has been neglected. Although the effect is small at this low Mach number, it still remains visible.

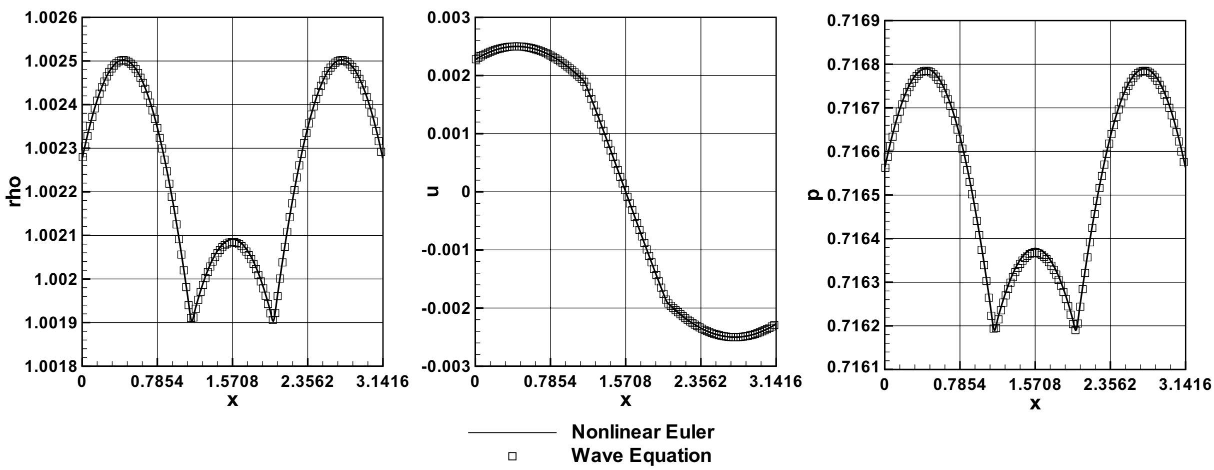


Fig. 2. The total flow quantities $\rho$, $u$ and $p$ at $t = 2.0$ of the boundary value problem (36) computed with the wave equation (34) compared with the non-linear, fully compressible Euler equations.

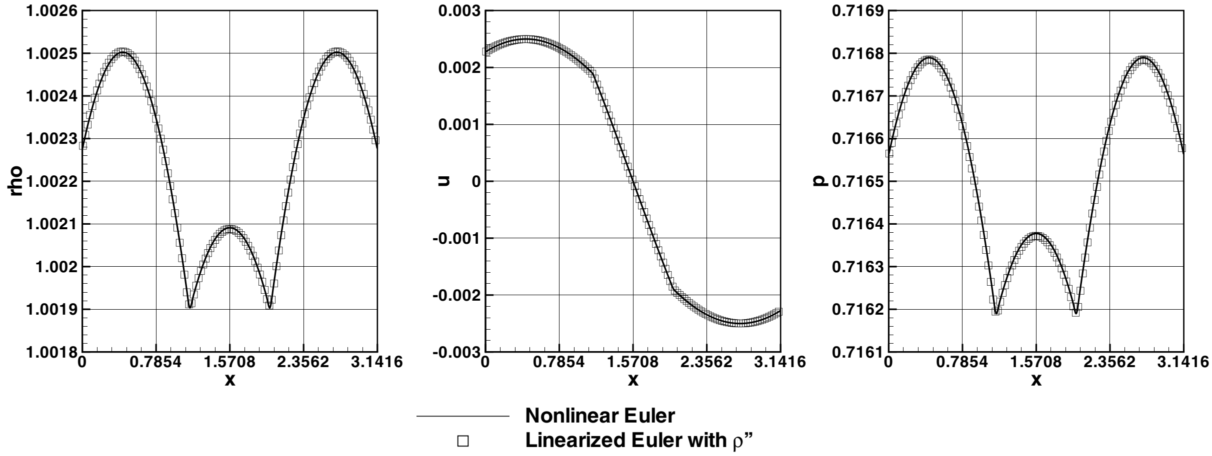


Fig. 3. The total flow quantities $\rho$, $u$ and $p$ at $t = 2.0$ of the boundary value problem (36) computed with the linearized Euler equations (46), (28) and (29) using the density perturbation quantity $\rho'' = \rho_2 + \rho'$, compared with the non-linear, fully compressible Euler equations.

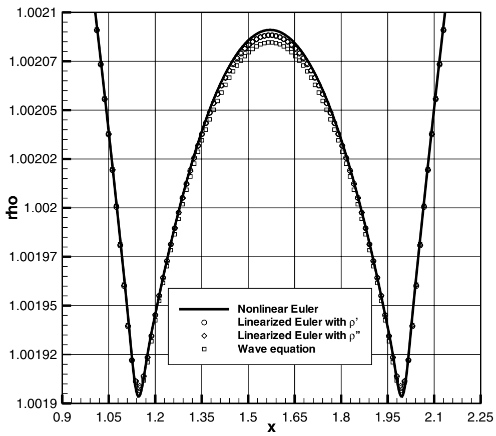


Fig. 4. Zoom into the total pressure $p$ at $t = 2.0$ of the boundary value problem (36). The reference solution obtained with the fully compressible Euler equations is compared with the results given by the linearized Euler equations (27)–(29) and the wave equation (34).

## 5. Conclusions

Motivated by asymptotic results for fluid flow at small Mach numbers we extended the expansion about incompressible flows to the more general situation, when heat conduction or compression from the boundary is active and cannot be neglected. The incompressible solution is replaced by the solution of the limit equations of the compressible Navier–Stokes equations, when the Mach number tends to zero. The usually applied additional assumptions of constant temperature and density are abolished. The state of expansion is given by a subsonic approximation of the compressible Navier–Stokes equations. We introduced multiple pressure variables, a thermodynamic background pressure and a hydrodynamic pressure term, to obtain the proper limit equations. The sum of the $M \to 0$ solution and perturbed quantities are inserted into the compressible equations to obtain evolution equations for the perturbations. A scaling with powers of the global Mach number gives a linearization of these equations by picking out the leading order terms and also identifies the main terms of the sources. We showed numerical results for a problem with compression from the boundary for which the $M \to 0$ equations have an analytic solution. The solution for $M \to 0$ plus the perturbations coincide very well with the numerical solution of the full Euler equations.

The considered perturbation approach is applicable to fluid flow with thermal processes. It is assumed in our considerations that these effects do not dominate the dynamics of the flow. We considered thermo-acoustic corrections of the $M \to 0$ solution. Otherwise the basic flow has to be described by the full compressible equations. Based on a multiple scale expansion with two space scales $\bar{x} = O(x)$, $\xi = O(Mx)$ and one time scale Klein [11] discovered in this case another consistent asymptotic expansion. Here, the pressure splits up into three main parts

$$p(x,t) = p_0(t) + Mp_1(\xi,t) + M^2 p_2(\bar{x},\xi,t) + \cdots, \tag{47}$$

where the pressure term $p_1$ is an acoustic pressure that only depends on the large-scale acoustic variable $\xi$. This corresponds to a regime where thermo-acoustic waves strongly influence the fluid flow, the order of the acoustic pressure in the expansion is even larger than that of the hydrodynamic pressure. This is no situation to apply a hybrid approach and to separate the flow from acoustics. The perturbation equations proposed in this paper do not cover this situation. Here, we assume that the fluid flow is always well described by the subsonic approximation. Within the construction of a numerical method for low Mach number flow Roller and Munz used the pressure variable $p_1$ to get an initial guess for the pressure iteration within a pressure-correction method in [14].

## Acknowledgment

This research was supported by Deutsche Forschungsgemeinschaft (DFG).

## References

[1] C. Bailly, C. Bogey, D. Juve, Computation of flow noise using source terms in linearized euler equations, AIAA J. 37 (1999) 409–416.
[2] H. Bijl, P. Wesseling, A unified method for computing incompressible and compressible flows in boundary-fitted coordinates, J. Comput. Phys. 141 (1998) 153–173.
[3] D.R. Chenoweth, S. Paolucci, Natural convection in an enclosed vertical air layer with large horizontal temperature differences, J. Fluid Mech. 169 (1986) 173–210.
[4] M. Dumbser, M. Käser, Arbitrary high order non-oscillatory finite volume schemes on unstructured meshes for linear hyperbolic systems, J. Comput. Phys. 221 (2007) 693–723.
[5] M. Dumbser, C.D. Munz, ADER discontinuous Galerkin schemes for aeroacoustics, CR Mécanique 333 (2005) 683–687.
[6] M. Dumbser, C.D. Munz, Building blocks for arbitrary high order discontinuous Galerkin schemes, J. Sci. Comput. 27 (2006) 215–230.
[7] R. Ewert, W. Schröder, Acoustic perturbation equation based on flow decomposition via source filtering, J. Comput. Phys. 188 (2003) 271–289.
[8] J.C. Hardin, D.S. Pope, An acoustic/viscous splitting technique for computational aeroacoustics, Theor. Comput. Fluid Dyn. 6 (1995) 323–340.
[9] S. Klainerman, A. Majda, Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, Commun. Pure Appl. Math. 34 (1981) 481–524.
[10] S. Klainerman, A. Majda, Compressible and incompressible fluids, Commun. Pure Appl. Math. 35 (1982) 629–653.
[11] R. Klein, Semi-implicit extension of a Godunov-type scheme based on low Mach number asymptotics I: one-dimensional flow, J. Comput. Phys. 121 (1995) 213–237.
[12] R. Klein, C.D. Munz, The multiple pressure variables (MPV) for the numerical approximation of weakly compressible fluid flow, in: K. Kozel, M. Feistauer, R. Rannacher (Eds.), Proceedings of Numerical Modelling in Continuum Mechanics, Charles University Prag, 1995.
[13] C.D. Munz, S. Roller, R. Klein, K.J. Geratz, The extension of incompressible flow solvers to the weakly compressible regime, Comput. Fluids 32 (2003) 173–196.
[14] S. Roller, C.D. Munz, A low Mach number scheme based on multi-scale asymptotics, Comput. Visual Sci. 3 (2000) 85–91.
[15] J.H. Seo, Y.J. Moon, Linearized perturbed compressible equations for low Mach number aeroacoustics, J. Comput. Phys. 218 (2006) 702–719.
[16] W.Z. Shen, J.N. Sorensen, Aeroacoustic modelling of low-speed flows, Theor. Comput. Fluid Dyn. 13 (1999) 271–289.
[17] S.A. Slimon, M.C. Soteriou, D.W. Davis, Computational aeroracoustics simulation using the expansion about incompressible flow approach, AIAA J. 37 (1999) 409–416.

[18] D.R. van der Heul, C. Vuik, P. Wesseling, A conservative pressure-correction method for the Euler and ideal MHD equations at all speeds, Int. J. Numer. Meth. Fluids 40 (2002) 521–529.
[19] D.R. van der Heul, C. Vuik, P. Wesseling, A conservative pressure-correction method for flow at all speeds, Comput. Fluids 32 (2003) 1113–1132.
[20] P. Wesseling, Principles of Computational Fluid Dynamics, Springer-Verlag, 2001.